# Unconditionally Stable MFLTD Method for the Full Wave Electromagnetic Simulation

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***Abstract*—A new unconditionally stable numerical method for the time domain simulation of electromagnetic problems using a combination of the Laguerre polynomials in the time domain and radial basis functions (RBF) in the spatial domain named mesh-Free Laguerre time domain (MFLTD) method is described. Unconditionally stable Laguerre based method may be computationally much more efficient than the FDTD methods and using RBFs allows computing the problems with complex-shapes in a non-uniform point's distribution. A numerical example at the final part of the communication shows the efficiency of the presented method in problems with sparse and coarse regions. Using the MFLTD method for this example, we can achieve a 96% reduction in the computation time and get an acceptable degree of accuracy in comparison to the conventional mesh-free time-domain (MFTD) method.**

***Index Terms*—Laguerre based scheme, Maxwell's equations, mesh-free Laguerre time domain (MFLTD) method, mesh-free (mesh-less) approach, unconditionally stable.**

## I. Introduction

For problems involving components with much smaller geometric features than the wavelengths of interest, the simulation of Maxwell's equations is time-consuming because of limits on the simulation time-step size. Recently, a new unconditionally stable scheme for simulation of the time-domain Maxwell's equations was introduced based on the Laguerre polynomials [1] to solve Maxwell's curl equations without stability limits. This method is a marching-on-in-degree method instead of marching-on-in-time and therefore the stability is no longer affected by the time step size [1]–[3]. The time step is used only to calculate the coefficients of source's Laguerre domain representation. As the transformation of the time-domain source to the Laguerre-domain is done only at the start of the computations, selecting a smaller value for $\Delta t$ can improves the accuracy of solution to a desired value without significant additional computation load. Therefore, Laguerre based method may be computationally much more efficient than the FDTD methods [1], [2].

On the other hands, mesh-free methods have shown to be more efficient than the traditional methods, like finite differences and finite element methods [4]–[8]. Because, these methods use only nodes to produce shape functions without connectivity information, i.e., the mesh. Moreover, the compact RBF and the domain decomposition method can provide a way to reduce the computational time and the ill-conditioning of the matrix in a mesh-free method [9], [10]. The mesh-free time-domain (MFTD) method for electromagnetic problems has been recently introduced which can improves the computational efficiency with respect to the conventional FDTD method [11]–[15]. Also, the ADI scheme has been used to obtain an unconditionally stable mesh-free method which can reduce the simulation time significantly [16]. However, the accuracy of ADI methods gets worse as the time step increases as the numerical dispersion error becomes larger [17].

This communication presents an unconditionally stable mesh-free Laguerre time-domain (MFLTD) method to have the advantages of Laguerre-based and mesh-free methods, simultaneously. In the next part, we describe the Laguerre domain representation of a function based on RBFs. This is follows by transforming Maxwell's equations to the Laguerre domain and presenting the MFLTD solution of them. Finally, we compare the MFLTD method with the MFTD method by a simulation example while a particular compact RBF is used.

Manuscript received March 03, 2011; revised September 25, 2011; accepted November 11, 2011. Date of publication March 02, 2012; date of current version May 01, 2012. This work was supported in part by the Iran Telecommunication Research Center (ITRC).

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Digital Object Identifier 10.1109/TAP.2012.2189710

## II. Basic Concepts

### A. Laguerre Transform of a Time Domain Function

An approximation of a function $u(t)$ may be written as a linear combination of $K$ modified Laguerre functions as [2],

$$u(t) = \sum_{p=0}^{K} u^p \psi_p(\bar{t}); \quad \bar{t} = s \cdot t \tag{1}$$

where $s$ is a scaling factor to increase the time scale to the order of second and $u^p$ is the $p$th expansion coefficient obtained by using the orthonormal property of the basis functions as,

$$u^p = \int_0^{\infty} \psi_p(\bar{t})\, u(\bar{t}) d\bar{t} \tag{2}$$

Then, the first-order partial derivative of function $u(t)$, with respect to the time, can be expressed as

$$\partial_t u(t) = s \sum_{p=0}^{K} u^{pp} \psi_p(\bar{t}) \tag{3}$$

where

$$u^{pp} = 0.5\, u^p + \sum_{k=0, p\neq 0}^{p-1} u^k \tag{4}$$

### B. Mesh-Free Method Based on RBF in the Spatial Domain

An approximation of a function $u(\mathbf{x})$ may be written as a linear combination of $M$ radial basis functions as [15],

$$u(\mathbf{x}) \simeq \sum_{i=1}^{M} \phi_i(\mathbf{x})\, u_i = \Phi(\mathbf{x})\mathbf{u}; \quad \mathbf{x} \in R^d \tag{5}$$

where $M$ is the number of data points, $\mathbf{x} = (x^1, x^2, \ldots, x^d)$ is the position vector, $d$ is the dimension of the problem, $\mathbf{u}^T = [u_1, u_2, \ldots, u_M]$ is the coefficient matrix to be determined, $\phi_i(\mathbf{x}) = \phi(\mathbf{x}, \mathbf{x}_i)$ is the radial basis function, and $\Phi(\mathbf{x}) =$

$[\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_M(\mathbf{x})]$. By defining the $M$-by-$M$ matrix $\Pi = [\Phi(\mathbf{x}_1); \Phi(\mathbf{x}_2); \ldots; \Phi(\mathbf{x}_N)]$ and forcing the interpolation to pass through all the $M$ collocation points $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M\}$, a solution for the coefficient matrix $\mathbf{u}$ is $\mathbf{u} = \Pi^{-1}\mathcal{U}$ and

$$u(\mathbf{x}) = \Phi(\mathbf{x})\mathbf{u} = P(\mathbf{x})\mathcal{U} \tag{6}$$

where

$$\begin{aligned} \mathcal{U}^T &= [u(\mathbf{x}_1), u(\mathbf{x}_2), \ldots, u(\mathbf{x}_M)], \\ P(\mathbf{x}) &= \Phi(\mathbf{x})\Pi^{-1}. \end{aligned} \tag{7}$$

The first-order partial derivative of the function $u(\mathbf{x})$, with respect to a space variables $\mathcal{X}$, can be expressed as

$$\partial_{\mathcal{X}} u^h(\mathbf{x}) = P_{,\mathcal{X}}(\mathbf{x})\mathcal{U}; \quad \mathcal{X} \in \{x^1, x^2, \ldots, x^d\} \tag{8}$$

where $P_{,\mathcal{X}}(\mathbf{x}) = \Phi_{,\mathcal{X}}(\mathbf{x})\Pi^{-1}$ and $\Phi_{,\mathcal{X}}(\mathbf{x}) = \partial\Phi(\mathbf{x})/\partial\mathcal{X}$.

## III. Maxwell's Equations in the Laguerre Domain Based on RBFs

Maxwell's equations characterize electromagnetic wave propagation completely, which can be written in a matrix form as [18],

$$\partial_t F(\mathbf{x}, t) = (A - B)F(\mathbf{x}, t) + J(\mathbf{x}, t). \tag{9}$$

where

$$\begin{aligned} F &= [\,E_x \quad E_y \quad E_z \quad H_x \quad H_y \quad H_z\,]^T, \\ J &= [\,J_x \quad J_y \quad J_z \quad 0 \quad 0 \quad 0\,]^T, \\ A &= \begin{bmatrix} \mathbf{0} & a_1 A_1 \\ a_2 A_2 & \mathbf{0} \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{0} & a_1 A_2 \\ a_2 A_1 & \mathbf{0} \end{bmatrix}, \end{aligned} \tag{10}$$

$a_1 = 1/(2\varepsilon)$, $a_2 = 1/(2\mu)$, and

$$A_1 = \begin{bmatrix} 0 & 0 & \partial_y \\ \partial_z & 0 & 0 \\ 0 & \partial_x & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \partial_z & 0 \\ 0 & 0 & \partial_x \\ \partial_y & 0 & 0 \end{bmatrix}. \tag{11}$$

In the above equations, $\partial_{\mathcal{X}}$ is the first derivative with respect to the $\mathcal{X} \in \{x, y, z, t\}$, $E$ is the electric field, $H$ is the magnetic field, $J$ is the total current density, $\varepsilon$ and $\mu$ are the electric permittivity and the magnetic permeability, respectively. Using (6) and then (1), an approximation of $F(\mathbf{x}, t)$ and $J(\mathbf{x}, t)$ in (9) can be written as a linear combination of $K$ modified Laguerre and $M$ radial basis functions as,

$$\begin{aligned} F(\mathbf{x}, t) &= P_{(6)}(\mathbf{x})\mathcal{F}(t) = P_{(6)}(\mathbf{x}) \sum_{p=0}^{K} \mathcal{F}^p \psi_p(\bar{t}), \\ J(\mathbf{x}, t) &= P_{(6)}(\mathbf{x})\mathcal{J}(t) = P_{(6)}(\mathbf{x}) \sum_{p=0}^{K} \mathcal{J}^p \psi_p(\bar{t}) \end{aligned} \tag{12}$$

where

$$\begin{aligned} \mathcal{F}(t) &= [F(\mathbf{x}_1, t), F(\mathbf{x}_2, t), \ldots, F(\mathbf{x}_M, t)], \\ \mathcal{J}(t) &= [J(\mathbf{x}_1, t), J(\mathbf{x}_2, t), \ldots, J(\mathbf{x}_M, t)], \end{aligned} \tag{13}$$

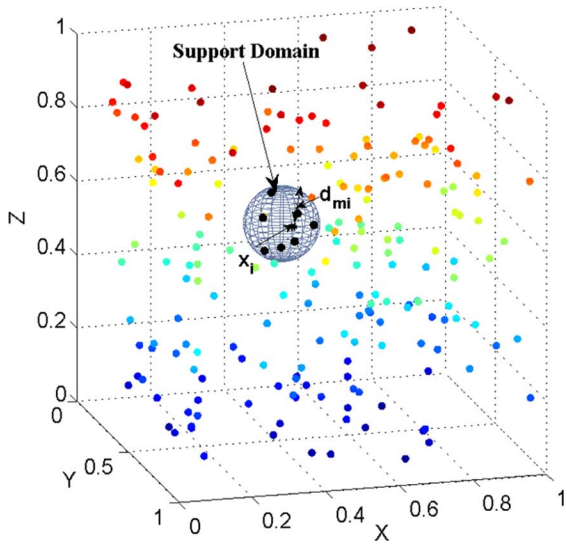


Fig. 1. The support domain for a point $\mathbf{x}_i$ in the mesh-free method.

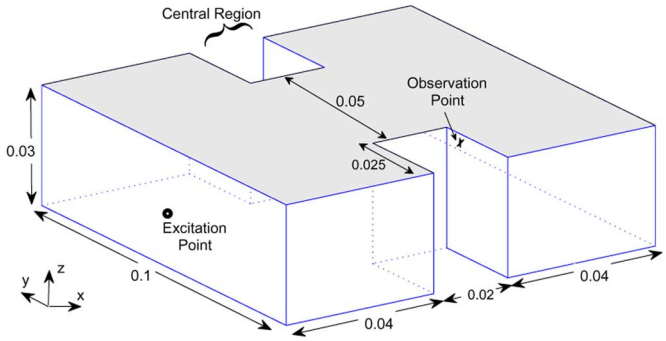


Fig. 2. Geometry of the simulated structure.

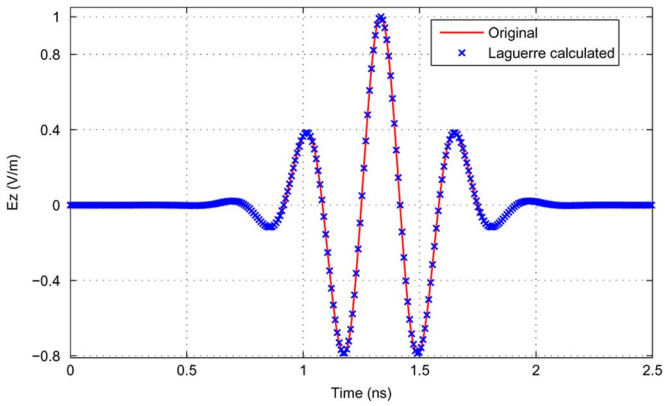


Fig. 3. Excitation pulse in the time domain.

$\mathbf{x} = x\hat{x} + y\hat{y} + z\hat{z}$ is the position vector,

$$\begin{aligned} \mathcal{F}^p &= \left[\mathcal{E}_x^p, \mathcal{E}_y^p, \mathcal{E}_z^p, \mathcal{H}_x^p, \mathcal{H}_y^p, \mathcal{H}_z^p\right]^T \\ \mathcal{J}^p &= \left[\mathcal{J}_x^p, \mathcal{J}_y^p, \mathcal{J}_z^p, 0, 0, 0\right]^T \end{aligned} \tag{14}$$

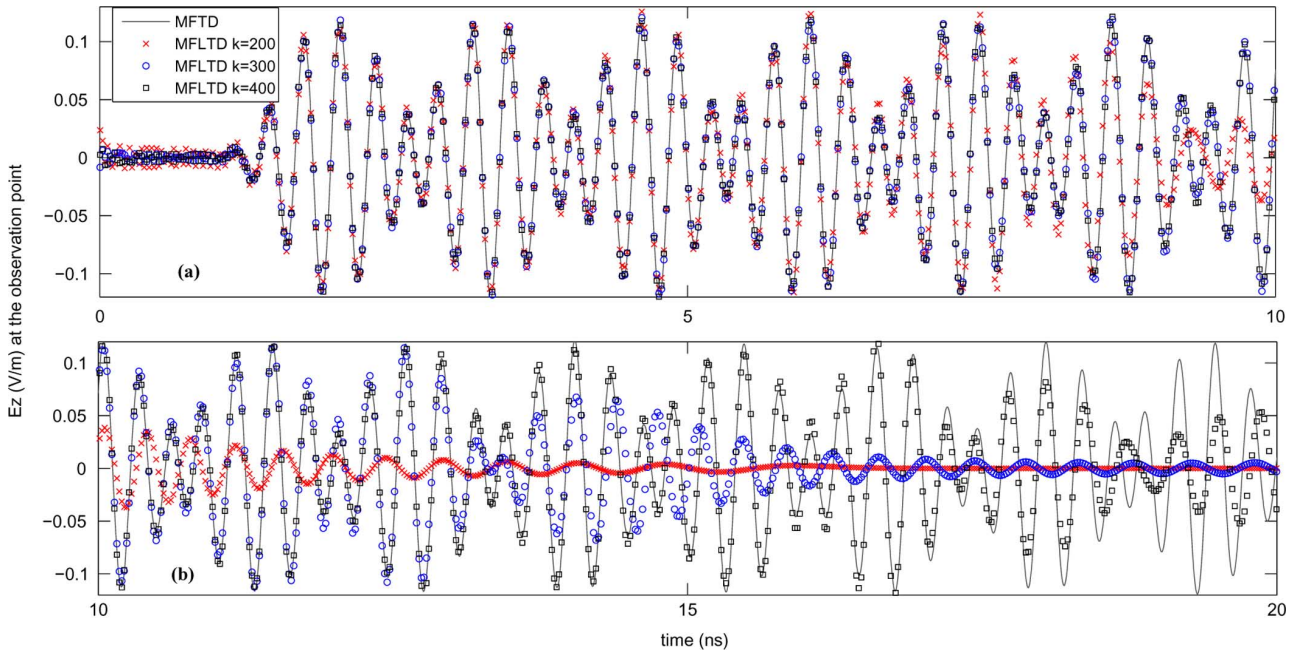


Fig. 4. $E_z$ at the observation point calculated by the MFTD and MFLTD with $K = 200$, 300, and 400 methods; (a) 0–10 ns and (b) 10–20 ns.

are $p$th unknown Laguerre coefficients at $M$ collocation points with $6 \times M$ elements, and $P_{(6)}$ is calculated from $P$ by six times repetition of each elements. Substituting (12) into (9), using (3), multiplying both side by $\psi_k(\bar{t})$ and integrating over $\bar{t} = [0, \infty)$ gives a set of $(K+1)$ boundary value problems (for $k = 0, 1, \ldots, K$) as,

$$sP_{(6)}\left(0.5\mathcal{F}^k + \sum_{d=0,k\neq 0}^{k-1} \mathcal{F}^d\right) = \left((A-B)P_{(6)}\right)\mathcal{F}^k + P_{(6)}\mathcal{J}^k. \quad (15)$$

Using (8) for the spatial derivatives of $A$ and $B$ in (15), the final 6 Maxwell's equations are written in the Laguerre domain as,

$$\mathcal{E}_a^k = \mathcal{P}_1\left(P_{,b}\mathcal{H}_c^k - P_{,c}\mathcal{H}_b^k\right) + \frac{2}{s}\mathcal{J}_a^k - 2\sum_{d=0,k\neq 0}^{k-1}\mathcal{E}_a^d \quad (16)$$

$$\mathcal{H}_a^k = \mathcal{P}_2\left(P_{,c}\mathcal{E}_b^k - P_{,b}\mathcal{E}_c^k\right) - 2\sum_{d=0,k\neq 0}^{k-1}\mathcal{H}_a^d; \quad (a,b,c) \in \{(x,y,z),(y,z,x),(z,x,y)\} \quad (17)$$

where $\mathcal{P}_1 = 2a_1(sP)^{-1}$ and $\mathcal{P}_2 = 2a_2(sP)^{-1}$. To reduce the required simulation memory and computation time, we can eliminate the unknown magnetic field components from (16), using (17) as,

$$\left\{1 - \mathcal{P}_1\mathcal{P}_2(P_{,b})^2 - \mathcal{P}_1\mathcal{P}_2(P_{,c})^2\right\}\mathcal{E}_a^k + \mathcal{P}_1\mathcal{P}_2P_{,b}P_{,a}\mathcal{E}_b^k + \mathcal{P}_1\mathcal{P}_2P_{,c}P_{,a}\mathcal{E}_c^k = \frac{2}{s}\mathcal{J}_a^k - 2\sum_{d=0,k\neq 0}^{k-1}\left(\mathcal{E}_a^d + \mathcal{P}_1P_{,b}\mathcal{H}_c^d + \mathcal{P}_1P_{,c}\mathcal{H}_b^d\right). \quad (18)$$

Consequently, the implicit relations for the electric fields can be written in the matrix form of,

$$\mathcal{C}\mathcal{E}^k = \mathcal{J}^k + \sum_{d=0,k\neq 0}^{k-1}\mathcal{G}(\mathcal{E}^d, \mathcal{H}^d); \quad k = 0, 1, \ldots, K \quad (19)$$

where $\mathcal{E}^k = [\mathcal{E}_x^k, \mathcal{E}_y^k, \mathcal{E}_z^k]^T$, $\mathcal{H}^k = [\mathcal{H}_x^k, \mathcal{H}_y^k, \mathcal{H}_z^k]^T$, $\mathcal{J}^k = [\mathcal{J}_x^k, \mathcal{J}_y^k, \mathcal{J}_z^k]^T$, and $\mathcal{G}$ is a linear function of $\mathcal{E}^d$ and $\mathcal{H}^d$. Even the $(3\times M\times M)$-by-$(3\times M\times M)$ coefficient matrix $\mathcal{C}$ in (19) is not sparse, but it is a constant matrix with respect to the coefficient number $k$. Therefore we need to perform the matrix inversion only once at the beginning of the computation. Starting from $k = 0$ and using the known coefficients of the current source, i.e., $\mathcal{J}^k$, the unknown coefficients $\mathcal{E}^k$ can be calculated by (19), recursively. Then, the magnetic field coefficients can be obtained from (17). Finally, the value of fields in a specific time and place is calculated from the above calculated coefficients, i.e., $\mathcal{E}^k$ and $\mathcal{H}^k$ at the collocation points and (12).

## IV. Simulation Results

Several choices are possible as RBF, e.g., multi-quadratics (MQ) or spline functions. Selecting the RBF type is based on the fact that it has good accuracy without extra load of computing the shape parameters which are needed in the other accurate approximations like the MQ [19]. On the other hand, using compactly supported RBFs (CS-RBF) could result in a sparse matrix, and thus the computational cost and memory requirements is reduced [9]. The comparative analysis of different CS-RBFs in [20] shows that Wu functions always converge in a smooth fashion. In this communication, the following CS-RBF proposed by Wu [10] is used,

$$\phi(r_i) = \begin{cases} (1-r_i)^4\left(4 + 16r_i + 12r_i^2 + 3r_i^3\right) & r_i \leq 1 \\ 0 & \text{elsewhere.} \end{cases} \quad (20)$$

where $r_i = \|\mathbf{x} - \mathbf{x}_i\|/d_{mi}$. Here, $d_{mi}$ represents the supported domain radius at the collocation node $\mathbf{x}_i$ as shown in Fig. 1. The selected Wu function in (20) is a $C^2$ continuous positive definite function on $\mathcal{R}^3$ with the proper convergence and number of terms [20].

To demonstrate the performance of the proposed approach, an $H$-shaped $0.1 \times 0.1 \times 0.03\,\mathrm{m}^3$ resonator with perfect electric conductor (PEC) walls is taken into consideration, as shown in Fig. 2 [15]. The resonator is excited with a z-directed and sine modulated Gaussian pulse $J_{sz} = e^{-f_0^2(t-t_d)^2} \cdot \cos(2\pi f_0(t-t_d))$ at [1.25, 5, 1.5] cm, of which center frequency is 3 GHz and time delay is 1.333 ns ($\lambda = 10$ cm). Fig. 3 compares the above pulse and an approximation of that with 100

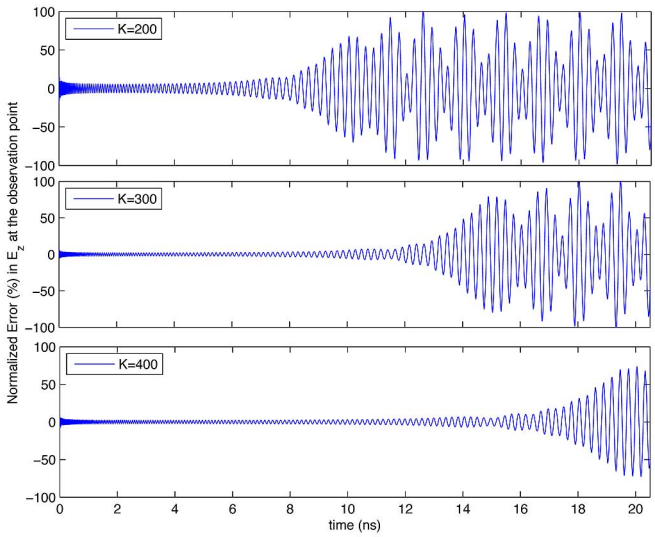


Fig. 5. Normalized error $(E_{z\mathrm{MFLTD}} - E_{z\mathrm{MFTD}})/\mathrm{Max}(E_{z\mathrm{MFTD}}) \times 100$ at the observation point.

Laguerre coefficients. We make an example non-uniform distribution points by selecting $\Delta = \lambda/40$ and $\lambda/20$ as the nodal spaces in the central and remaining regions, respectively. The distribution of $E$ and $H$ field nodes are similar to the Yee's method while the PEC boundaries are modeled by the $E$ field nodes. The simulation is done for 20 ns and $E_z$ is accumulated at an observation point of [1.25, 5, 8.75] cm, considering the number of Laguerre coefficients, $K$, equal to 200, 300, and 400 and scaling factor $s = 5 \times 10^{10}$, as shown in Fig. 4. Selection of $K$ is dependent on the final time of the simulation, $t_f$. For example an accurate data up to 10 ns needs at least 300 coefficients (Fig. 4). The simulation times are approximately 0.74, 1.5, 3, and 130 seconds for the MFLTD with $K = 200, 300, 400$ and the MFTD methods, respectively. Using the recursive algorithm presented at the end of previous section, the memory usage of MFTD and MFLTD methods are approximately equal and independent of $K$. The normalized error of MFLTD method with respect to the MFTD method in calculation of $E_z$ at the observation point is shown in Fig. 5. The error is less than 10% for $K = 400$ and $t_f = 15$ ns while the computation time is reduced by 96%. A PC with a E8500 processor and 4 GB RAM was used in these simulations.

## V. Conclusion

We have proposed an unconditionally stable method for simulation of the time domain Maxwell's equations. The mesh-free scheme in the spatial domain and Laguerre-based approach in the time domain get the ability of a very fast simulation of complex shapes to the presented method. In an example, a 96% reduction in the simulation time is achieved while the normalized error is less than 10% in comparison to the explicit mesh-free method. The presented formulation is for the general case and any kind of basis function can be used in the mesh-free part of the method. Possible extensions of this research are to implement the absorbing boundary condition, like PML, in the MFLTD method for the simulation of open boundaries and to consider adaptive data nodes and overlapping domain decomposition method for further reduction of the CPU time.

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