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Proposing the Frequency Dependent Shape Functions for Meshless Method in Electromagnetics

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ABSTRACT — **The root of this study, comes from the lack of regularity where exists in shape function selection in direct meshless methods. In this work, we are going to establish a technique which is based on both analytical and simulated results; helps select a well-behaved shape function for which the shape parameters have been predetermined. This work has been focused on wave equation as the most important equation in electromagnetics. Comparing the results with conventional meshless radial point interpolation method (CRPIM), finite element method (FEM) and the method of moments (MOM), the technique shows extremely good accuracy despite a great reduction in time consumption of selecting a shape function.**

Index Terms **— Direct meshless methods, partition of unity, shape functions, Shepard's weighting functions, wave equation.**

I. INTRODUCTION

Meshless method is a novel numerical technique to overcome the problems that still exist on the way of numerical methods [1]. This method can avoid the construction of tedious and difficult meshes and is also applicable on problems that need to be approximated, nonlinearly. Its accuracy can be more than FEM, because the nodes are scattered arbitrary and the elements and the limitations that they impose to a given problem are omitted [1]. However, one of the major problems that is the time consumption of weak form numerical methods still exists. Recently, some direct meshless methods have been proposed to solve the time consumption problem [2-3]. The direct and conventional (indirect) meshless methods are different just in construction of the shape functions, while the direct meshless method proposes the shape functions, directly [2-3]. This novel method caused a great reduction in time consumption because of canceling the first inversion step and improved the accuracy by proposing an analytical shape function which its derivatives exist analytically [3]. Up to now, a trading off method was used in compatibility process of direct shape functions that shows the lack of strict technique by which the compatible shape function and its shape factors are predetermined [3]. Consequently, there is a question which has not been answered yet: beside partition of unity method (PUM) condition [4] and bell-shaped form functions, which other condition must be satisfied when a shape function is proposed directly? That's what we talk about. Each shape function has some controlling parameters called *shape factors*. These factors control the decay and overhanging behavior of the shape function to make it compatible on a given PDE problem [1]. In this work, a technique is proposed based on the mathematical analysis and linear algebra and is combined with simulation results. Finally, we have introduced a shape function based on the technique for a given PDE (wave equation) that shows really good results.

II. MATHEMATICAL CONCEPTS

A. Approximation of the Solution Function

Consider an operator equation as

$$L\varphi = g \tag{1}$$

The solution of (1) in three dimensional cases can be approximated as follows

$$\varphi_{app}(x,y,z) = \sum_{i=1}^{n} N_i(x,y,z)\varphi_i \tag{2}$$

In which, n denotes the number of scattered nodes in problem domain Ω , $N_i(x,y,z)$ is the shape function at node ith and φ_i is the value of solution function at this node [1].

B. Partition of Unity

In engineering texts, it is usually ignored to discuss on the origin of the approximated solution (2). The origin what imposes some constraints on the shape functions $N_i(x,y,z)$. In conventional (indirect) meshless methods, the approximation or basis functions construct the shape functions. So, shape functions automatically satisfy these constraints known as PUM property [4]. However, in direct meshless methods due to proposing the shape functions directly, it is necessary for the shape functions to satisfy PUM condition [3]. As will be seen, the PUM helps control the *maximum error* of approximation function. Let to summarize them here. If Ω_i be an open cover called patch presented in Fig. 1, $N_i(x,y,z)$ must satisfy the following conditions [4]

$$\begin{aligned} &\text{supp}(N_i) \subset \text{closure}(\Omega_i) \\ &\textstyle\sum_i N_i = 1 \text{ on } \Omega \\ &\|N_i\|_{L_\infty(R^n)} \le C_0 \\ &\|\nabla N_i\|_{L_\infty(R^n)} \le C_1 \\ &\varphi_{app} \subset H^1(\Omega) \end{aligned} \tag{3}$$

The first condition means that N_i has nonzero value only in Ω_i . In other words, it is locally supported. C_0 and C_1 are two constants. $L_\infty(R^n)$ and $H^1(\Omega)$ refer to Lebesgue and Hilbert spaces, respectively. Under these conditions known as PUM condition, the PUM theorem presented in [4] is expressed as follows by some changes in expression based on [5].

Theorem 2.1 The maximum error of approximation function with respect to the exact solution and also that of their derivatives are predictable and possible to control.

Proof: See [4].

This theorem is why we want to be forced with the PUM. The expression of Hilbert spaces shows that the fifth condition is always held for the practical solutions of physic equations. It is due to the properties of the shape functions mentioned in (3), i.e. locally supported, differentiability and this fact that φ_i is always finite in practical cases. Here, the first part of the technique is established as

1) Shape function must satisfy the first four PUM conditions given in (3).

C. Finite Difference Approximation

Although the approximation solution (2) is derived in the functional sense but it is also a solution of (1). It means that (2) must be able to satisfy (1), directly. So, substituting (2) into (1) yields

$$L(\sum_i N_i(x,y)\varphi_i) = g \quad (4)$$

or

$$(\sum_i \varphi_i L N_i(x,y)) = g \quad (5)$$

We claim that *under a given frequency*, the shape function should approximately hold the behavior of solution function. This claim will be proven by the aid of simulation results in the last section. The word "approximately", intentionally entered in the expression to stress on this fact that a special behavior of the solution function which is the *maximum and minimum* of its finite difference computational molecule, is under consideration. Compatibility of each shape function on a given computational molecule comes from the fact that the operator imposes some common constraints as the order of differentiability (smoothness) to both solution and shape function. One of the properties of this compatibility is that the coefficient (stiffness) matrix in meshless method takes the form of coefficient matrix in FDM [7]. So it becomes sparser and less time is needed to calculate its inverse. This matching also brings the Kronecker delta property for shape functions. This property simplifies the imposition of boundary conditions. Therefore, the instant results of the claim are used to improve the other parts of technique as follows

2) Shape function should be differentiable, at least up to the order of its operator (smooth enough).

3) Shape function should have the same maximum and minimum values such as computational molecule of its PDE.

III. Proposing a Shape Function Using The Given Technique for Wave Equation

A. Definition of the Problem

The problem under consideration is illustrated in Fig. 2 where an electromagnetic wave is incident upon an imperfectly conducting circular cylinder, having a smooth, convex cross section. We employ the so-called on surface radiation condition (OSRC) method to solve for scattering by the cylinder [7]. According to [7] the governing equation is as follows

$$\frac{\partial^2 \varphi^{sc}}{\partial s^2} + 2j(\frac{j}{\rho} - k_0)\left[jk_0(1+\tau) + \frac{1}{2\rho} - \frac{j}{8\rho^2(\frac{j}{\rho}-k_0)} \right]\varphi^{sc} = -2j(\frac{j}{\rho}-k_0)\left[jk_0\tau\varphi^{inc} - \frac{\partial \varphi^{inc}}{\partial \rho}\right] \quad (6)$$

or

$$\frac{\partial^2 \varphi^{sc}}{\partial s^2} - g\varphi^{sc} = -q \quad (7)$$

in which $\tau = \eta$ for magnetic polarization and $\tau = \eta^{-1}$ for electric polarization. The $\varphi^{inc} = \exp(-jk_0\rho\cos\theta)$ is the incident wave. Here, θ is the angular coordinate in cylindrical system, ρ is the radius and η is the normalized impedance on scatterer surface. According to the electric or magnetic polarization, φ^{inc} will be E_z or H_z. Using the functional (weak form) of wave equation as

$$f(\varphi_z) = \frac{1}{2}\int_\Omega \left[(\frac{\partial \varphi_z}{\partial x})^2 - k_0{}^2 g\varphi_z{}^2 + q\varphi_z \right] dx \quad (8)$$

and substituting (2) into the functional, the following system of equations is achieved

$$[K]_{n\times n}[\varphi]_{n\times 1} = [B]_{n\times 1} \quad (9)$$

where $[K]_{n\times n}$ is the stiffness (coefficient) matrix and $[B]_{n\times 1}$ is the excitation one.

B. Shepard's Method

According to somewhat proposed by Shepard in scattered data fitting [8], shape function must consist of *two* functions. The former controls the distance compatibility and the next one controls the direction effects. In some special cases in which the solution is scalar and no direction effect is appeared, the second function will also control the distance and causes more degree of freedom by which the accuracy will increase.

In Shepard's method, a simple rule is also considered by which the second PUM condition in (3) is satisfied that is

$$N_i(x) = \frac{w_i(x)}{P(x)} \quad (10)$$

where w_i is called weighting functions and

$$P(x) = \sum_i w_i(x) \quad (11)$$

We use the multiplication of two functions $B_i(x)$ and $C_i(x)$ to construct the shape function $N_i(x)$ in one dimensional i.e.

$$w_i = B_i \times C_i \quad (12)$$

$$B_i(x) = \exp(-\alpha r_i{}^2)$$
$$C_i(x) = \cos(\beta\pi r_i) \quad (13)$$
$$r_i = x - x_i$$

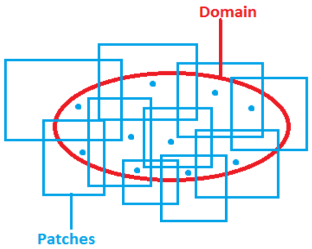

Fig. 1. The problem domain and patches.

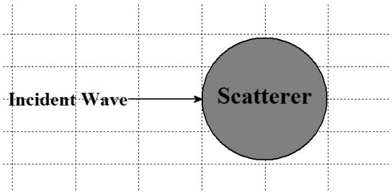

Fig. 2. The scatterer and incident wave.

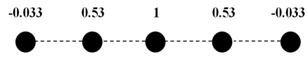

Fig. 3. The computational molecule of fundamental wave equation.

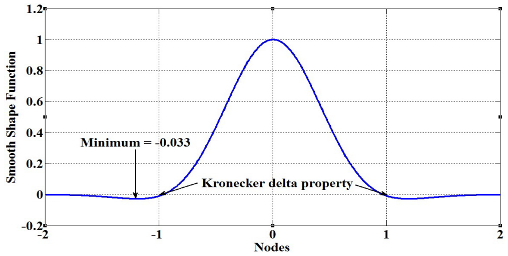

Fig. 4. Technical shape function for the scattering wave equation.

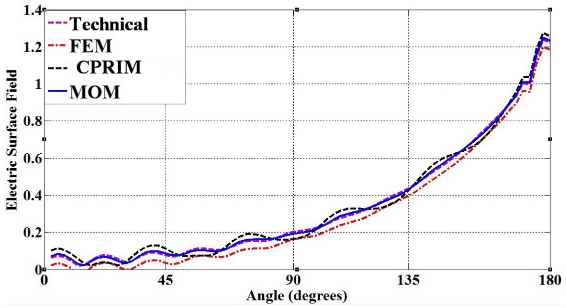

Fig. 5. Electric surface field on scatterer.

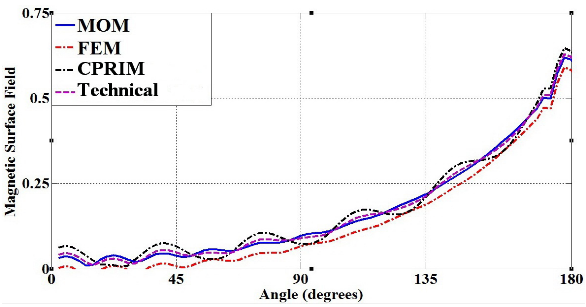

Fig. 6. Magnetic surface field on scatterer.

where α and β are positive constant values. These constants will control the *decay and overhanging* behavior of the shape function to be matched on the different parts of technique. Furthermore, due to considering the shape functions as the building or fundamental blocks of the solution function, it leads us toward matching the shape function on computational molecule of the fundamental wave equation e.g. free space wave equation. This computational molecule has been shown in Fig. 3. Also, the technical shape function can be seen in Fig. 4.

IV. Comparison with Other Methods

Here, accuracy of the proposed shape function based on the technique is compared with FEM, CRPIM with multiquadric (MQ) approximation function [1] and MOM (as the exact solution) [7]. Figs. 5 and 6 show the accuracy of proposed technique for scattered field on the surface.

V. Conclusion

In this work, a technique has been proposed to select the shape functions in direct meshless methods. This approach helps to overcome the problem of irregularity and time consumption of the shape function compatibility in direct meshless methods. A shape function that completely matches on the criterion is proposed and testing in a given electromagnetic problem, showed extremely good agreement to exact solution, in comparison with a number of other numerical methods.

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