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Criterion for selecting the shape functions in electromagnetic meshless methods

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Abstract: The root of this study, comes from the lack of regularity where exists in shape function selection in meshless methods. Up to now, shape functions in meshless methods were being established by basis functions. In other words, there was no predictable rule by which the shape functions be predetermined, without any need to basis functions. In presented approach, the authors are going to establish a criterion, which is based on both analytical and simulation results; helps select an approximately well-behaved shape function for which the shape parameters have been predetermined. This work focused on both Laplace and wave equations as the most important equations in electrostatic and electromagnetic problems to show the validation of the proposed approach in computational electromagnetics. Comparing the results with radial point interpolation method as the most common existing meshless method, finite-element method and finite-difference method, the criterion shows extremely good accuracy despite a great reduction in time consumption rate for selecting a compatible shape function.

1 Introduction

Meshless method is a novel numerical technique to overcome the problems that still exist on the way of numerical methods [1]. Meshless methods are similar to finite-element method (FEM) such that in both of them, basis functions are proposed and the shape functions are constructed by basis functions in a matrix inversion step that evaluates the shape functions, numerically. Then, a system of equations is constructed and usually solved by another matrix inversion process. However, in comparison with FEM [2], the meshless method can avoid the construction of tedious and difficult meshes and is also applicable to problems that need to be approximated by non-polynomial basis functions. Its accuracy can be more than FEM, because the nodes are scattered arbitrary. Furthermore, elements and limitations that they impose to a given problem are cancelled [1]. However, one of the major problems that is high computational cost, still exists in meshless method. Recently, some direct meshless methods have been proposed to solve the time consumption issue [3–7]. The direct and conventional (indirect) meshless methods are different just in construction of the shape functions, whereas the direct meshless method proposes the shape functions, directly [3–7]. This novel method caused a great reduction in time consumption, because of cancelling the first matrix inversion calculation step and improved the accuracy by proposing an analytical shape function, which its derivatives exist, analytically [7]. Nevertheless, the direct meshless method has been using a trading off technique in process of selecting the shape parameters that show the lack of strict criterion by which the shape function and its shape factors be predetermined [3, 7]. Consequently, there is a question that has not been answered yet: which criterion or regularity must be satisfied when a shape function is proposed directly in the novel method? That's what we talk about. It is well known that most of the shape functions have the bell-shaped configuration and each shape function has some controlling parameters called shape factors [8–17]. These factors control the decay and overhanging behaviour of the shape function to make it compatible on a given partial differential equation (PDE) problem [1]. In this work, a criterion is proposed for predetermining the shape factors based on the mathematical analysis and is combined with simulation results. Finally, we have introduced some direct shape functions based on the criterion for two given PDEs in electromagnetics, that is, Laplace's and wave equations. It will be shown that the criterion gives really good results in comparison with other numerical methods. This achievement leads us towards the future approaches for imposing the criterion to time-domain problems [18–23].

The rest of the paper has been organised as follows. At first, we establish the mathematical concepts of our approach in different subsections. We start with the Ritz's method in variational analysis. Then, the partition of unity (PUM) theorem is explained and the direct shape functions based on finite-difference method (FDM) will be introduced. We show that Shepard's method helps us select some good shape functions. After mathematical concepts, we check the criterion in both electrostatic and electrodynamic cases, that is, Laplace and wave equation. Finally, comparing results of criterion-based method and some other numerical methods show really good agreements and accuracy for the criterion.

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2 Mathematical concepts

In this section, we try to establish the mathematical foundations of our criterion. At first, it is necessary to have a glance on variational principle. Next subsection expresses the PUM and its importance using a theorem. In a lemma, we will prove that solutions of all practical electromagnetic problems belong to Hilbert space by which the PUM can control the error of approximation solution. Next, we focus on the relation between shape functions and FDM computational molecule. This novel relation constructs the main idea of the criterion. Finally, Shepard's weighting functions help propose the shape functions, analytically.

2.1 Variational weak form and meshless method

The weak form method in a short expression is a method, which replaces a PDE by its corresponding integral called functional. This functional is usually solved by Ritz's or Galerkin's method. The construction of functional in calculus of variations is similar in both FEM and the meshless method based on the following inner product definition [1]. For two functions φ and ψ, the inner product is defined as

$$\langle \varphi,\ \psi \rangle = \int_{\Omega} \varphi \bar{\psi} \mathrm{d}\Omega \tag{1}$$

where $\bar{\psi}$ denotes the complex conjugate of ψ. Consider an operator equation as

$$L\varphi = w \tag{2}$$

where φ needs to be determined and w is a known excitation function. The Ritz's method is a variational method, in which (2) is formulated in terms of an inner product called functional. Then, the solution of the PDE can be obtained by minimising the functional given by

$$\frac{1}{2}[\langle L\tilde{\varphi},\ \tilde{\varphi}\rangle - \langle \tilde{\varphi},\ w\rangle - \langle w,\ \tilde{\varphi}\rangle] \tag{3}$$

with respect to $\tilde{\varphi}$, where $\tilde{\varphi}$ denotes the approximation function [1]. The $\tilde{\varphi}$ in above relation will be expressed as

$$\tilde{\varphi}(x) = \sum_{i=1}^{n} N_i(x)\varphi_i \tag{4}$$

in which n denotes the number of scattered nodes in the problem domain Ω. The N_i is the chosen shape function or expansion function and φ_i is the value of solution function at node i that must be determined. After substituting (4) into (3) for minimising, its partial derivatives with respect to φ_i must be forced to vanish to give the minimum points.

The operator L is self-adjoint if

$$\langle L\varphi,\ \psi\rangle = \langle \varphi,\ L\psi\rangle \tag{5}$$

It can be shown if L is self-adjoint, Galerkin's method results in the same system of equations as those given by the Ritz's method [2].

2.2 PUM theorem

As mentioned, in some special cases such 'rough' changes of the solution, the polynomial approximation cannot model the behaviour of solution function well enough and 'non-polynomial' functions are required. In general, it is very difficult if not impossible to enforce interelement continuity on non-polynomial approximation functions [24]. Here is the importance of PUM that offers a means, which holds the local approximation properties when the approximation functions are non-polynomial. In fact, PUM eliminates the tedious constraint of interelement continuity and replaces it with a new constraint as 'locally supported' for each approximation function. In conventional (indirect) meshless methods, the basis functions construct the shape functions and shape functions automatically satisfy the PUM property [1]. However, in direct meshless methods because of direct presentation of shape functions, it is necessary for the shape functions to satisfy the PUM theorem [24]. Let us summarise it here.

Definition 2.1: If Ω_i be an open cover called patch presented in Fig. 1, the maximum number of overlapping patches at an arbitrary point x must be a finite number M named pointwise overlap condition. Under this condition, the following conditions must be satisfied by N_i [24]

$$\mathrm{support}(N_i) \subset \mathrm{closure}(\Omega_i) \tag{6}$$

$$\sum_i N_i = 1,\ \text{on } \Omega \tag{7}$$

$$\|N_i\|_{L_\infty(R^n)} \le C_0 \tag{8}$$

$$\|\nabla N_i\|_{L_\infty(R^n)} \le C_1 \tag{9}$$

$$\tilde{\varphi} \subset H^1(\Omega) \tag{10}$$

The first condition means that N_i has non-zero value only in Ω_i. In other words, it is locally supported. C_0 and C_1 are two constants. L_∞ and H^1 refer to Lebesgue and Hilbert spaces, respectively. The Hilbert space is defined as

$$H^1 = \{f | f \text{ and } f' \in L_2\} \tag{11}$$

where L_2 is the space of finite-energy functions. Under these conditions known as Lipschitz conditions, the PUM theorem presented in [24] is expressed as follows by some changes in expression.

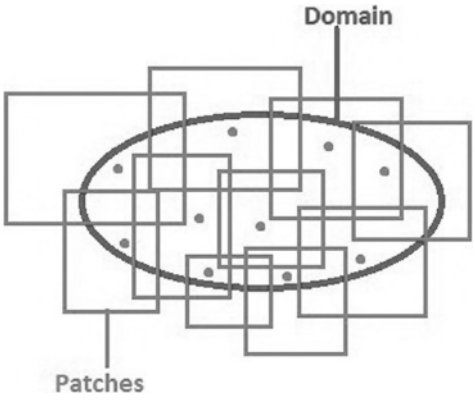

Fig. 1 *Problem domain and patches*

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Theorem 2.1: The $\|\text{error}\|_{L_2}$ of $\tilde{\varphi}$ with respect to the exact solution and also that of their derivatives are predictable and possible to control by factors M, C_0 and C_1, that is,

$$\|\varphi - \tilde{\varphi}\|_{L_2} \leq \sqrt{M} C_0 \left(\sum_i \mathrm{e}_0^2(i) \right)^{\frac{1}{2}} \tag{12}$$

$$\|\nabla(\varphi - \tilde{\varphi})\|_{L_2} \leq \sqrt{2M} \left(\sum_i \left[C_1^2 \mathrm{e}_0^2(i) + C_0^2 \mathrm{e}_1^2(i) \right] \right)^{\frac{1}{2}} \tag{13}$$

where $e_0(i)$ and $e_1(i)$ are the errors of approximation function and its derivative at ith patch.

Proof: see [24].

This theorem is why we want to be forced by the PUM when the approximation spaces are not necessarily of polynomial functions. Let us focus on the condition in (10) with more resolution using a lemma to better illustrate its application and answer to this question that: is (10) depending on the shape function when the previous conditions (6)–(9) are satisfied?

Lemma 2.1: The condition in (10) is always held under the first four conditions in (6)–(9) at practical physics equations.

Proof: Let check the condition in (10) by taking the following Lebesgue integrals on both shape function and its derivative to see if the integrals exist or not. These integrating come from the definition of H^1. We have

$$\begin{aligned} \int_\Omega |\tilde{\varphi}|^2 \mathrm{d}\Omega &= \int_\Omega \left| \sum_i N_i(\Omega) \varphi_i \right|^2 \mathrm{d}\Omega \\ &= \int_\Omega \left| \sum_i \sum_j N_i(\Omega) N_j(\Omega) \varphi_i \varphi_j \right| \mathrm{d}\Omega \\ &\leq \int_\Omega \left(\sum_i \sum_j \left| N_i(\Omega) N_j(\Omega) \right| |\varphi_i \varphi_j| \right) \mathrm{d}\Omega \\ &= \sum_i \sum_j |\varphi_i \varphi_j| \left(\int_\Omega |N_i(\Omega) N_j(\Omega)| \, \mathrm{d}\Omega \right) \end{aligned} \tag{14}$$

In case of the derivative

$$\begin{aligned} \int_\Omega |\tilde{\varphi}'|^2 \mathrm{d}\Omega &= \int_\Omega \left| \sum_i N_{i'}(\Omega) \varphi_i \right|^2 \mathrm{d}\Omega \\ &= \int_\Omega \left| \sum_i \sum_j N_{i'}(\Omega) N_{j'}(\Omega) \varphi_i \varphi_j \right| \mathrm{d}\Omega \\ &\leq \int_\Omega \left(\sum_i \sum_j |N_i'(\Omega) N_j'(\Omega)| |\varphi_i \varphi_j| \right) \mathrm{d}\Omega \\ &= \sum_i \sum_j |\varphi_i \varphi_j| \left(\int_\Omega |N_i'(\Omega) N_j'(\Omega)| \mathrm{d}\Omega \right) \end{aligned} \tag{15}$$

where in two-dimensional (2D) cases the last term in (15) is

$$\begin{aligned} \sum_i \sum_j |\varphi_i \varphi_j| \int_x \int_y \left(\left| \frac{\partial N_i(x,y)}{\partial x} \frac{\partial N_j(x,y)}{\partial x} \right| \right. \\ \left. + \left| \frac{\partial N_i(x,y)}{\partial y} \frac{\partial N_j(x,y)}{\partial y} \right| \right) \mathrm{d}x \mathrm{d}y \end{aligned} \tag{16}$$

The last parts of (14) and (15) show that the condition in (10) is held, if both integral and $\varphi_i \varphi_j$ exist. Owing to the properties of shape functions mentioned in (6)–(9), that is, the nature of locally supported and differentiability of N_i, the integral is always finite. On the other hand, solutions of practical physics equations are always finite. Hence, the last terms in (14) and (15) are always finite and (10) is always held under the first four conditions in (6)–(9). This knowledge is important when the shape functions are introduced directly, and shows that under (6)–(9), (10) does not affect the shape functions in practical cases. Hence, for proposing a direct shape function, only the conditions in (6)–(9) must be considered. Here, the first term of the criterion is established as

1. 'Shape function must satisfy the first four Lipschitz conditions given in (6)–(9)'.

2.3 Finite difference approximation

Although the approximation solution (4) is usually used in the functional sense, but it is also a solution of (2). It means that (4) must be able to satisfy (2), directly. So, substituting (4) into (2) yields

$$L\left(\sum_i N_i \varphi_i \right) = w \tag{17}$$

or

$$\sum_i \varphi_i L N_i = w \tag{18}$$

At this point, we claim that the shape function should approximately hold the behaviour of solution function. The word 'approximately', intentionally entered in the expression to stress on the fact that a special behaviour of the solution function, which is its finite difference computational molecule, is under consideration. Compatibility of each shape function on a given computational molecule comes from the fact that the operator imposes some common constraints as the order of differentiability (smoothness) to both solution and shape functions. Hence, to solve any PDE, we are interested in those kinds of shape functions that besides satisfying the general Lipschitz conditions, take the form of computational molecule. Now, we must specify that which kind of the behaviour of computational molecule should be imitated by shape functions? The extremum points are the answer. For this purpose, the 'maximum' and 'minimum' of the shape function should be equal to that of the computational molecule of its PDE. One of the properties of this compatibility is that the coefficient (stiffness) matrix in meshless method takes the form of coefficient matrix in FDM. So, it becomes sparser and less time is needed to calculate its inversion. After all of these steps, just one

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question still exists: how should the overhanging of a shape function become compatible? To answer based on our experiments, the shape function should hold the 'Kronecker' delta property, that is,

$$N_i\left(r_j\right)=\begin{cases}1, & \text{if } r_i=r_j,\ i,\ j=1,\ 2,\ \ldots,\ n\\ 0, & \text{if } r_i\neq r_j\end{cases} \tag{19}$$

The Kronecker delta property [1] simplifies the imposition of boundary conditions and this is the reason that it has been entered in our expression as a part of the ongoing criterion. Experiments showed that the first zero of shape function must lie between its first maximum and minimum points. All parts of claim will be shown in the figures related to shape functions at following sections. Therefore the results of the claim are used to improve the criterion as follows:

1. Shape function should be differentiable, at least up to the order of its operator (smooth enough).
2. Under the Kronecker delta property, the extremum points of each shape function should be matched on that of the computational molecule of its PDE.

One may ask that in those problems with discontinuity, it would be better for shape functions not to be differentiable near discontinuities to better illustrate the solution function. So, sounds like that there is a contrast between second part of criterion and the mentioned fact. Here, it must be noted that one way to solve a discontinuity problem is to split it into two continuous problem and impose the continuity condition on the edge of discontinuity. This paper uses this method to solve the problems. Here, the criterion becomes complete, and in the next subsection we introduce some techniques by which the criterion be satisfied.

2.4 Shepard's method

According to somewhat proposed by Shepard in scattered data fitting; shape function must consist of 'two' functions [25]. The former controls the distance compatibility and the latter controls the direction effects. In some special cases that the solution is scalar and no direction effect is appeared, the second function will also control the distance and causes more degree of freedom by which the accuracy will increase. In Shepard's method, a simple rule is also considered by which the second Lipschitz conditions in (7) are satisfied, that is, [25]

$$N_i(x,\ y)=\frac{w_i(x,\ y)}{P(x,\ y)} \tag{20}$$

where

$$P(x,\ y)=\sum_i w_i(x,\ y) \tag{21}$$

In fact, we should introduce the weighting function w_i to reach the shape function N_i. The subscript i refers to the weighting function of the ith node. So, (20) guarantees that the summation of all shape functions at an arbitrary point is equal to one. We use the multiplication of two functions named $E_i(x,\ y)$ and $D_i(x,\ y)$ to construct the weighting function as

$$w_i(x,y)=E_i(x,y)D_i(x,y) \tag{22}$$

E_i usually is of the exponential type such that be able to control the decay of the shape function and D_i is usually of the cosine or polynomial type to control its overhanging behaviour.

3 Laplace equation and static shape functions

In this section, we start with the functional principle of the Laplace equation. Then we propose a shape function that is based on the different parts of the criterion. Finally, a given Laplace problem with Dirichlet boundary conditions is solved and the accuracy of criterion will be proved, numerically.

3.1 Laplace's equation and its weak form

Let be with Laplace's equation as an important and self-adjoint operator equation for electric potential to better illustrate the proposed approach, given below

$$\begin{aligned}\nabla^2\varphi&=0\\ \varphi(x,\ 0)&=0\\ \varphi\left(x_{\max},\ y\right)&=0\\ \varphi\left(x,\ y_{\max}\right)&=10\\ \varphi(0,\ y)&=0\end{aligned} \tag{23}$$

As seen, the Dirichlet kind boundary condition is considered on four boundaries of the problem. We consider the 2D cases of this equation for which $x_{\max}=y_{\max}$. According to the functional principle, the functional of Laplace equation is as follows

$$F(\varphi)=\frac{1}{2}\int_\Omega\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right]\mathrm{d}\Omega \tag{24}$$

Substituting (4) into (24) and taking derivative with respect to φ_i, gives

$$\frac{\partial F}{\partial\varphi_i}=\sum_{j=1}^{M}\varphi_j\int_\Omega\left[\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right)+\left(\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)\right]\mathrm{d}\Omega \tag{25}$$

that according to the Ritz's method, must be equated to zero to satisfy the stationary point condition. As seen, the variational principle only handles the first-order derivative of the shape function and confirms that the space of solution functions is H^1. The system of equations is constructed as

$$[K]_{M\times M}[\phi]_{M\times 1}=[B]_{M\times 1} \tag{26}$$

The elements of stiffness matrix $\boldsymbol{K}$, $\boldsymbol{k}_{ij}$ and excitation matrix $\boldsymbol{B}$, $\boldsymbol{b}_i$ are as follows

$$\boldsymbol{k}_{ij}=\int_\Omega\left[\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right)+\left(\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)\right]\mathrm{d}\Omega \tag{27}$$

and

$$\boldsymbol{b}_i=0 \tag{28}$$

Then, the Dirichlet boundary condition at the lth node is

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imposed, directly into matrices $\boldsymbol{K}$ and $\boldsymbol{B}$ using the penalty method as [2]

$$\boldsymbol{k}_{ll} = 10^{70} \text{ and } \boldsymbol{b}_l = b_0 \times 10^{70} \tag{29}$$

where b_0 is the Dirichlet boundary value (here $b_0 = 10$). We suggest E_i and D_i for constructing the weighting function in (22) as follows

$$E_i(r) = \exp\left(-\alpha r_i^2\right) \tag{30}$$

$$D_i(r) = \cos\left(\pi\beta r_i\right) \tag{31}$$

where

$$r_i = \sqrt{\left(x - x_i\right)^2 + \left(y - y_i\right)^2} \tag{32}$$

(x_i, y_i) is the position of scattered node and α and β are positive constants called shape factors. Substituting w_i into (20), the derivative of proposed shape function with respect to x can be obtained as

$$\frac{\partial N_i}{\partial x} = \frac{\left(\partial w_i/\partial x\right)P - \left(\partial P/\partial x\right)w_i}{P^2} \tag{33}$$

where

$$\frac{\partial w_i}{\partial x} = -2\alpha\left(x - x_i\right)w_i - E_i(\beta\pi)^2\left(x - x_i\right)\operatorname{sinc}\left(\beta r_i\right) \tag{34}$$

and

$$\frac{\partial P}{\partial x} = \sum_i \frac{\partial w_i}{\partial x} \tag{35}$$

Derivative with respect to y is similar to (34) and (35), in which $(x - x_i)$ has been replaced by $(y - y_i)$. The function's compatibility onto the criterion is controlled by α and β.

3.2 Compatibility on the first and second terms of criterion

The proposed shape function N_i is compactly supported. Owing to the Shepard's method, it also satisfies the second PUM condition in (7). The third and fourth conditions are also held, because $\|N_i\|_{L_\infty} \leq 1$ and $\|\nabla N_i\|_{L_\infty} \leq C_1$. The constant C_1 is a simulation result and will be determined in the compatibility process of the third term. N_i is also differentiable up to the second order.

3.3 Compatibility on the third term of criterion

As it was mentioned in Section 2, under the Kronecker delta property, the extremum points of N_i should be similar to the FDM computational molecule of desired PDE, which in this case is the Laplace equation. Here, we are going to calculate the extremum points of the shape function, using the finite difference method. By using the fourth-order finite difference approximation of the Laplace equation for electric potential, we can achieve the computational molecule of the shape function as [26]

$$N_i^j = 0.27\left(N_{i+1}^j + N_{i-1}^j + N_i^{j+1} + N_i^{j-1}\right) - 0.017\left(N_{i+2}^j + N_{i-2}^j + N_i^{j+2} + N_i^{j-2}\right) \tag{36}$$

More attention discovers this fact that the absolute maximum of every shape function and computational molecule is equal to one. Hence, we should focus on other maximum or minimum points. The computational molecule of (36) has been shown in Fig. 2*a*. As seen, the absolute maximum is 1 and the absolute minimum is −0.017. Then, the Kronecker delta property must be satisfied. Finally, the shape parameters of (30) and (31) are chosen as $\alpha = 1.4$, $\beta = 0.52$ and C_1 would be equal to 1.45. Fig. 3 illustrates the obtained shape function and its derivative in two-dimensions related to one of the middle nodes in an uniform node distribution (6 × 6). Here, the accuracy of the proposed shape function based on the criterion is compared with FDM, conventional radial point interpolation method (RPIM) with multi-quadrics basis function [1] and previous irregular direct proposed shape function in [7]. Fig. 4*a* shows the accuracy according to the following error definition

$$\text{error} = \frac{1}{M}\frac{\sum_i^M \left|\varphi_{\text{exact}}^i - \varphi_{\text{numerical}}^i\right|}{\sum_i^M \left|\varphi_{\text{exact}}^i\right|} \tag{37}$$

where M is the total number of scattered nodes, $\varphi_{\text{numerical}}^i$ and φ_{exact}^i are the numerical and the analytical solutions at the ith node, respectively. As expected, the accuracy has been improved using the criterion-based shape function and confirms that the criterion is valid in electrostatic problems. Fig. 4*b* shows the computational time of different methods for the Laplace equation, when runs on an Intel Core i3-2.26 GHz processor.

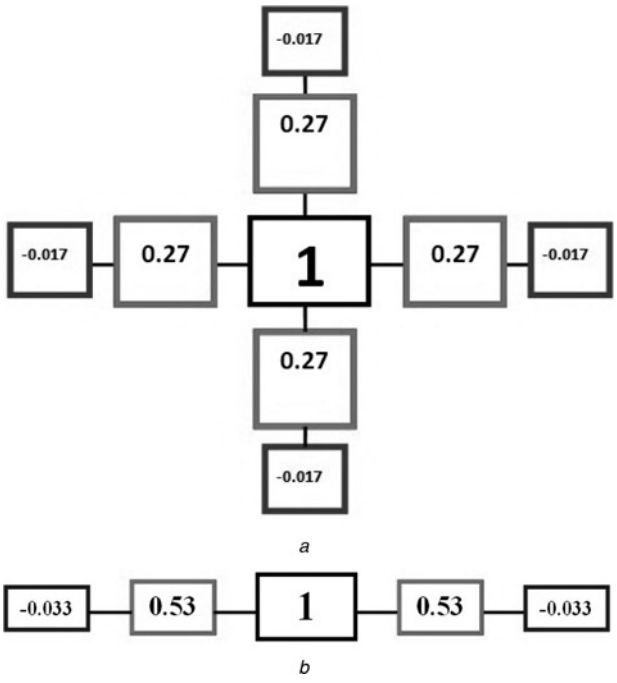

Fig. 2 *Computational molecule*

a Laplace equation
b Wave equation

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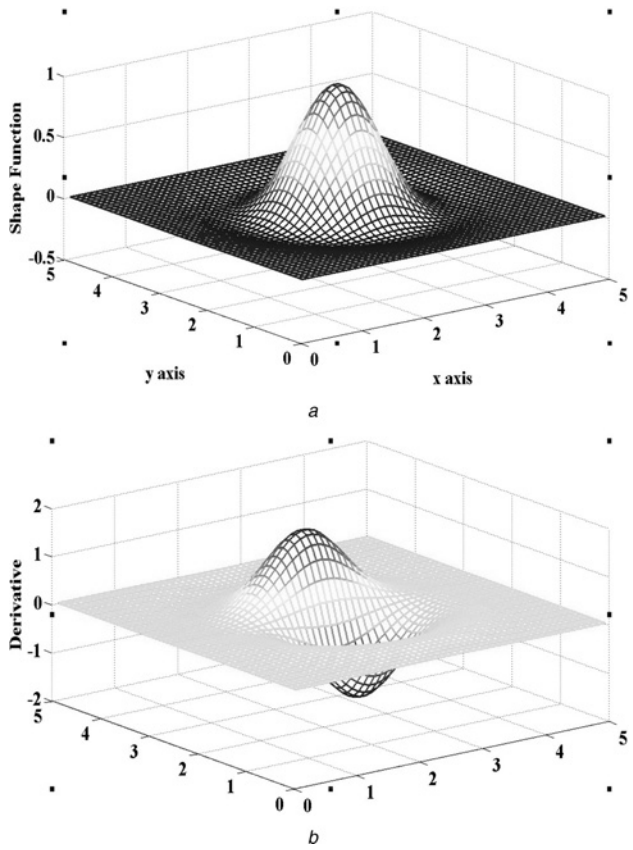

Fig. 3 *Criterion-based shape functions*

a Laplace equation
b Its derivative

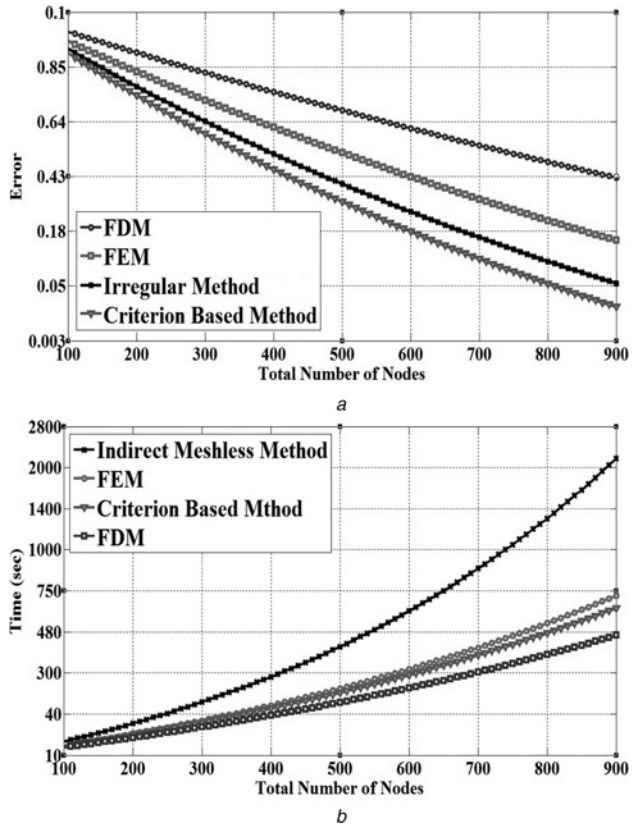

Fig. 4 *Error and computational time of Laplace problem*

a Error of criterion-based meshless method in comparison with other methods
b Computational time of the criterion-based and other methods

4 Wave equation and dynamic shape functions

In this section, we are going to check the criterion in a more complicated problem such as an electromagnetic problem. The first subsection introduces the functional principle of the wave equation. Next, we make the shape function compatible on the criterion. An interesting case is that the shape function is frequency dependent. Finally, the accuracy of the criterion is studied on the desired electromagnetic problem.

4.1 Wave equation and its weak form

To confirm the criterion, strictly, we need to check it in a more complicated situation, such as wave equation in an inhomogeneous medium given below [2]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f\frac{\mathrm{d}\phi_z}{\mathrm{d}x}\right)+k_0^2\,g\varphi_z=0 \tag{38}$$

where f and g are coefficient functions, determined later. The problem under consideration is illustrated in Fig. 5, where a uniform plane wave is incident on an inhomogeneous dielectric slab backed by a conducting surface [2]. The dielectric has thickness d, relative permittivity ε_{r} and relative permeability μ_{r}, both of which can be function of x. The surrounding medium is free space. We are interested in finding the power reflected by the slab. For this example, d is set equal to $5\lambda_0$ and the following complicated medium (known as shielding dielectrics) parameters are considered for dielectric

$$\mu_{\mathrm{r}}=2-0.1j \tag{39}$$

$$\varepsilon_{\mathrm{r}}=4+\mu_{\mathrm{r}}\left(1-\frac{x}{L}\right)^2 \tag{40}$$

It is well known that any plane wave can be decomposed into an E_z-polarised plane wave for the electric field and a H_z-polarised plane wave [2]. Now, let us first identify the domain for the analysis. The domain of the problem is obviously semi-infinite, but the meshless method cannot be applied to such an unbounded domain. Therefore we need to reduce the domain by introducing an artificial boundary with an appropriate boundary condition. According to [2],

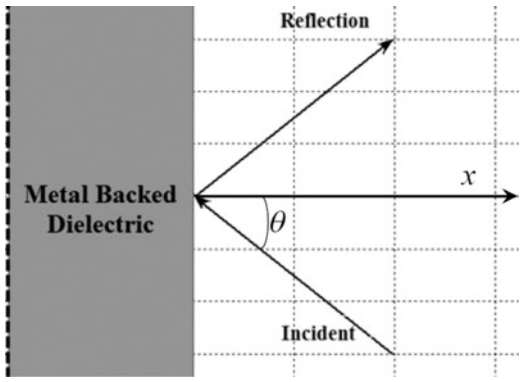

Fig. 5 *Problem domain of the incident wave on a metal backed dielectric*

Length of dielectric is d

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this boundary condition can be expressed for E_z-polarised as

$$\left[\frac{\mathrm{d}E_z(x)}{x}+\mathrm{j}\boldsymbol{k}_0\cos\theta E_z(x)\right]_{x=d_i}=2\mathrm{j}\boldsymbol{k}_0\cos\theta E_0\mathrm{e}^{\mathrm{j}\boldsymbol{k}_0 x\cos\theta} \quad (41)$$

where d_i is d and $1.33d$ for the split domain. The artificial boundary condition for magnetic polarisation can be given by a same relation. Hence, we consider a general form for this type of boundary condition as

$$\frac{\partial\varphi_z}{\partial x}+h\varphi_z=v\varphi_0 \quad (42)$$

where φ_0 is the wave amplitude, h and v are derived according to (41). Now, it is time to separate two different polarisations below and express the boundary conditions of each polarisation. Then, each of these polarisations would be solved under the criterion using direct meshless method.

1. 'Electric polarisation'

For E_z-polarisation, it can be shown that the Helmholtz equation (38) governs the electric field E_z when its parameters are set as follows [2]

$$\begin{aligned}&\varphi_z=E_z,\quad \varphi_0=E_0\\ &f=\frac{1}{\mu_\mathrm{r}},\quad g=\varepsilon_\mathrm{r}-\frac{1}{\mu_\mathrm{r}}\sin^2\theta\end{aligned} \quad (43)$$

For absorbing boundary condition, there are following parameters for (42)

$$h=\mathrm{j}\boldsymbol{k}_0\cos\theta,\quad v=2\mathrm{j}\boldsymbol{k}_0\cos\theta\,\mathrm{e}^{\mathrm{j}\boldsymbol{k}_0 d_i\cos\theta} \quad (44)$$

and the essential boundary conditions such as Dirichlet and continuity conditions must be enforced as

$$E_z(0,y)=0,\ E_z(d,y)=E_z(d,y) \quad (45)$$

1. 'Magnetic polarisation'

For H_z-polarisation, (38) governing the magnetic field H_z with the following parameters [2]

$$\begin{aligned}&\varphi_z=H_z,\quad \varphi_0=H_0\\ &f=\frac{1}{\varepsilon_\mathrm{r}},\quad g=\mu_\mathrm{r}-\frac{1}{\varepsilon_\mathrm{r}}\sin^2\theta\end{aligned} \quad (46)$$

For absorbing boundary condition, similar to the previous one, we have

$$h=\mathrm{j}\boldsymbol{k}_0\cos\theta,\quad v=2\mathrm{j}\boldsymbol{k}_0\cos\theta\,\mathrm{e}^{\mathrm{j}\boldsymbol{k}_0 d_i\cos\theta} \quad (47)$$

and the continuity condition must be forced as

$$H_z(d,y)=H_z(d,y) \quad (48)$$

There is also a Neumann condition for magnetic polarisation as

$$\frac{\partial H_z(0,y)}{\partial x}=0 \quad (49)$$

It must be noted that (49) is a natural condition that is automatically satisfied.

The functional of wave equation in one-dimensional (1D) case is as follows [2]

$$F(\varphi)=\frac{1}{2}\int_\Omega\left[f\left(\frac{\partial\varphi}{\partial x}\right)^2-\boldsymbol{k}_0^2\,g\varphi^2\right]\mathrm{d}x+\frac{1}{2}\left[h\varphi^2-2v\varphi\right]_{x=d_i} \quad (50)$$

Substituting (4) into (50) in 1D case and taking derivative with respect to φ_i, gives

$$\frac{\partial F}{\partial\varphi_i}=\sum_j^M\varphi_j\int_\Omega\left[f\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right)-\boldsymbol{k}_0^2 g N_i N_j\right]\mathrm{d}\Omega \quad (51)$$

That is equated to zero to satisfy the stationary point condition (Ritz's method). The system of equations is constructed as

$$[\boldsymbol{K}]_{M\times M}[\phi]_{M\times 1}=[\boldsymbol{B}]_{M\times 1} \quad (52)$$

The elements of stiffness matrix $\boldsymbol{K}$ and excitation matrix $\boldsymbol{B}$ are as follows

$$\boldsymbol{k}_{ij}=\int_\Omega\left[f\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right)-\boldsymbol{k}_0^2 g N_i N_j\right]\mathrm{d}\Omega \quad (53)$$

and

$$\boldsymbol{b}_i=0 \quad (54)$$

The Dirichlet boundary condition is imposed as mentioned in (29) and third-kind boundary condition, that is, (42), is imposed as

$$\boldsymbol{k}_{ll}=\boldsymbol{k}_{ll}+h(d_i) \quad (55)$$

$$\boldsymbol{b}_l=v(d_i)\varphi_0 \quad (56)$$

where l denotes nodes under third kind boundary condition, that is, the nodes at d and $1.33d$.

4.2 Compatibility on the third term of criterion

At this point, we are going to construct the FDM approximation of the fourth order for the free space wave equation as [26]

$$N_i=\frac{1}{30-(k_0\Delta x)^2}\left[16\left(N_{i+1}+N_{i-1}\right)-\left(N_{i+2}+N_{i-2}\right)\right] \quad (57)$$

where Δx denotes the spatial increment in FDM. Unlike the static case, FDM computational molecule and hence the shape function of the wave equation are frequency dependent.

We use the free space wave equation, because the shape function is locally supported and completely normalised function (its absolute maximum is equal to 1), so it can be considered as the 'building' or 'fundamental block' of each solution function. On the other hand, in numerical methods variable permittivity and permeability at each subdomain is usually approximated by constant values. From this point of view, in such cases as wave equation, the shape function

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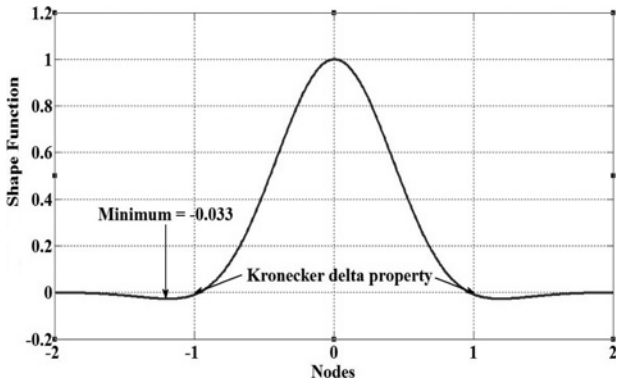

Fig. 6 *Proposed shape function for wave equation at $f = 3$ GHz and $\Delta x = 0.01$*

could be matched on the computational molecule of 'fundamental' equation of that type, that is, the equation in which both permittivity and permeability are constants at entire domain such as the free space wave equation. It will be shown that this idea gives very good result even in very complicated mediums with discontinuities.

The general form of proposed shape function for this example is similar to previous one according to (30) and (31), and just values of shape factors are changed to make it compatible on the computational molecule of the wave equation. So, we just focus on the derivation of new factors. Since the frequency-dependent coefficients in computational molecule of the wave equation, the 'bell-shape' behaviour of a shape function may be lost in some frequencies. Here, due to prevent this loss, the following condition must be held

$$30 - \left(\boldsymbol{k}_0 \Delta x\right)^2 > 16 \tag{58}$$

that yields

$$\Delta x < 0.6\lambda_0 \tag{59}$$

Now, let us analyse the problem at frequency $f = 3$ GHz and $\Delta x = 0.01$. The related computational molecule is shown in Fig. 2*b*. The proposed shape function matched on the criterion has been illustrated in Fig. 6. Note that the first zero has been lied between the first maximum and minimum points. To check the validity of criterion, we are interested in the evaluation of reflection coefficient defined as [2]

$$R = \frac{\varphi_z(d) - \varphi_0 \mathrm{e}^{\mathrm{j}\boldsymbol{k}_0 d \cos\theta}}{\varphi_0 \mathrm{e}^{-\mathrm{j}\boldsymbol{k}_0 d \cos\theta}} \tag{60}$$

Here, the proposed method has been compared with the analytical solution of reflection coefficient, the FEM and finally, the previous irregular direct proposed shape function solution in [7]. Fig. 7 shows the accuracy of the proposed method in both electric and magnetic cases. As seen, the accuracy has also improved using the criterion in dynamic (frequency dependent) equations. The figures of reflection coefficients illustrate that at larger incident angles, all compared methods have better results.

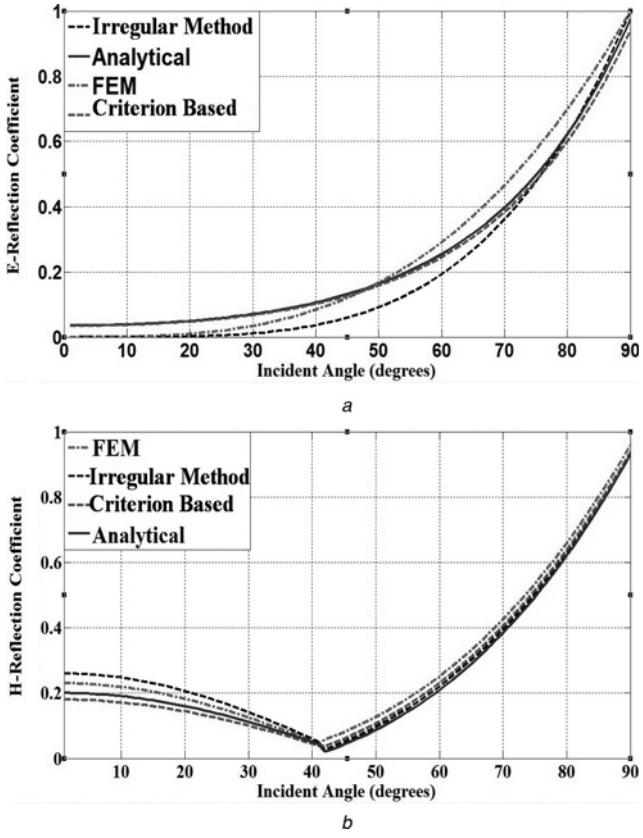

Fig. 7 *Reflection coefficients from horizontal to vertical incident*
a *E*-polarisation
b *H*-polarisation

5 Conclusions

In this work, a criterion was proposed to select the shape functions in meshless methods. This approach helps overcome the problem of irregularity and time consumption of the shape function selection in meshless methods. The origin of this criterion comes from the Lipschitz conditions and FDM theory that imposes some common properties to both the solution and shape functions. Some shape functions that are completely matched on the criterion were proposed and testing in the given electrostatic and electrodynamic problems, showed extremely good agreement to the exact solutions, in comparison with a number of other numerical methods.

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