A Modified Wavelet-Meshless Method for Lossy Magnetic Dielectrics at Microwave Frequencies

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In this work, the use of existing multiresolution analysis (MRA) in meshless method is studied to analyze some dielectrics for which the permeability and permittivity are complex at microwave frequencies. It will be shown that the existing MRA has some disadvantages and non meaningful aspects when is used in above dielectric simulation. In general, this type of MRA belongs to other areas of study such as image processing. Specifying these aspects and proposing some modifications to overcome the disadvantages of existing MRA for reaching a strict method of using wavelets in area of high frequency dielectrics, is the aim of this paper. We have selected the meshless method that is one of the newest and most powerful existing numerical methods as an area of using the modified MRA called computational MRA (CMRA). The use of CMRA in meshless methods, not only leaves most of the disadvantages, but also is faster and more accurate in comparison with the most other numerical methods.

***Index Terms*—Complex dielectrics, Daubechies' wavelets, meshless method, multiresolution analysis (MRA), partition of unity, scaling functions, shape functions, Shepard's method.**

I. INTRODUCTION

THE simulation of complex mediums, such as dielectrics in the electromagnetic controversy, plays an important role to discover the nature of materials at microwave frequencies. So, using some efficient computational techniques such as a combination between wavelet and meshless methods will discover more aspects of the wave behavior propagated through dielectrics.

The radial point interpolation method (RPIM), is the most common meshless approach in computational electromagnetics (and for simplicity we name it just meshless method). Similar to other numerical methods, it is based on approximation of the solution function of partial differential equations (PDEs). This approximation is according to those shape functions or interpolation functions that are able to model the solution, well enough. Up till now, many shape functions have been used to achieve the mentioned purpose and all of them are constructed by some basis functions such as multiquadrics (MQ), Gaussian (EXP), thin plane spline (TPS) and the logarithmic (LOG) basis functions [1]. Either weak or strong forms of meshless methods can solve a PDE using an approximation solution function expressed as a summation of the finite number of shape functions. In fact, the concept of approximation is nothing more than a finite summation of functions that can model the solution function in a good manner. Because the multiresolution analysis (MRA) theory also proposes the approximation of solution function by a finite summation of special functions known as the scaling-wavelet functions, it can be used in some rare aspects of the electromagnetic meshless methods [2]–[5]. Actually, scaling-wavelet functions have the same role as the shape functions in the wavelet-meshless methods. However, as mentioned, the existing application of MRA in meshless method is just an approximating role, without any need to do some modifications on the solution function. In other words, we are interested in the solution of PDE with all of its aspects, either noise and abrupt changes or well and smooth behavior. The present approximations based on MRA in meshless methods, use some theorems and concepts of MRA whose applications are usually in data compression, noise cancelation, etc. This direct employment in computational electromagnetics and complex dielectrics, combines some irrelevant concepts such as the orthogonal complement spaces or Daubechies' method [6] for constructing the scaling-wavelet functions and does not pay attention to the following two basic properties of the shape functions in meshless methods [1].

1) The nature of shape function in the meshless methods is a nonlinear function with even symmetry and bell-shaped form that usually holds the Kronecker delta property (Fig. 2) [1]. However, most of the conventional scaling-wavelet functions do not hold these properties in general (such as those in Daubechies' method [6]).
2) The order of continuity for the functions in the additional approximation terms used for modeling the discontinuities in meshless methods should be one (sharpness) (Fig. 2) [1]. However, the conventional wavelet functions do not necessarily obey this fact (such as those in Daubechies' method [6]).

These two reasons are the experimental results reported independently in different meshless method courses [1] and cause the two following major shortcomings:

1) high rate of time consumption in calculation process;
2) inability of achieving a highly accurate approximation.

In this work, a modified MRA called *computational* MRA (CMRA) in which some modifications are applied on the use of the scaling-wavelet functions [6] as the shape functions for the meshless methods is proposed.

This paper is organized as follows. In Section II, we express the mathematical concepts of the meshless methods, the present MRA's approximation theory in the expression of the meshless methods [2]–[5] and also its disadvantages that lead us toward CMRA. Section III proposes the CMRA in a complete sense. Finally, in Section IV, ability of the proposed CMRA in solving the PDEs of electromagnetic dielectrics will be investigated.

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II. Mathematical Concepts

This section is started with general concepts of the meshless method. Then, we move toward introducing the theorem of scaling functions in existing MRA. Finally, we show the disadvantages of the conventional MRA applied to the meshless methods which cause it to fail as a highly accurate and fast computational method.

A. Approximation Function in Meshless Method

Consider an operator equation as

$$Lu = w \tag{1}$$

where u needs to be determined and w is a known excitation function. The approximation function $\tilde{u}$ is expressed as

$$\tilde{u}(x) = \sum_{i=1}^{n} N_i(x) u_i \tag{2}$$

in which n denotes the number of scattered nodes in the problem domain Ω. N_i is the chosen shape function or interpolation function and u_i is the value of solution function at node i that must be determined. This paper uses the above approximation function in weak form meshless methods.

B. Theorem of Scaling Function Properties

At this point, we are going to present the properties of scaling function to propose a desired MRA for a given PDE. The following theorem in this subsection presents the properties of φ.

Theorem 2.1: If the scaling function φ satisfies the following conditions [6]:

$$\text{support}(\varphi(x-k)) \subset \text{closure}(\Omega_k) \tag{3}$$

$$\varphi(x-k) \text{ be continuous on } \Omega_k \tag{4}$$

$$\int_{\Omega_k} \varphi(x-k)\, dx = 1 \tag{5}$$

$$\int_{\Omega_k} \varphi(x-k)\varphi(x-l)\, dx = \delta_{kl}^{\text{Kronecker}} \tag{6}$$

$$\varphi(2^j x - l) = \sum_{k \in Z} p_{k-2l} \varphi(2^{j+1} x - k) \tag{7}$$

then, $\{V_j, j \in Z\}$ is a MRA where V_js are the multiscale approximation spaces [6]. In fact, (7) shows the refinement equation. In each level of resolution, the approximation function of MRA is similar to (2) in which N_i has been replaced by φ_i as the scaling function [2]–[5]. The φ and also ψ (wavelet function) are usually constructed in an indirect manner using an iteration process. The Daubechies method is the most common technique for this construction [6]. But as mentioned, this method does not hold the two basic properties of shape functions in meshless method.

III. CMRA Method

This section expresses the partition of unity method (PUM) and its necessity in view of meshless method. The PUM forces some constraints on the shape functions used in the meshless methods. Then, we are going to modify the PUM by proposing two different functional approximation spaces; one for modeling the smooth behavior of the solution function and another for the sharp behavior. Afterwards, the CMRA will combine the properties of scaling and shape functions to reach a highly accurate computational method in area of electromagnetics. Finally, Shepard's method will realize this idea.

A. Smooth and Sharp Shape Functions

So far, the properties of scaling functions in view of MRA were expressed. However, as previously mentioned, the final purpose of this paper is to use the MRA in meshless methods. So, the knowledge of scaling functions is just a half of the picture. The next half is to express the basic properties of shape functions in meshless method. The PUM considers some properties for shape functions such as those in Theorem 2.1 for scaling functions. As indicated by [7], PUM helps control the rms value of error in meshless methods. If we are interested in a highly accurate multi scale method, all aspects of both scaling and shape functions should be considered; this is what we name CMRA. It will be shown that CMRA combines the properties of scaling and shape functions according to Theorem 2.1 (in view of MRA) and PUM (in view of meshless method). Before modification, let summarize the PUM.

Definition 3.1: If Ω_i is an open cover called patch presented, the maximum number of overlapping patches at an arbitrary point x must be a finite number M named pointwise overlap condition. Under this condition, the following conditions must be satisfied by N_i or shape function [7]

$$\text{support}(N_i(x)) \subset \text{closure}(\Omega_i) \tag{8}$$

$$\sum_i N_i(x) = 1, \text{ on } \Omega \tag{9}$$

$$\|N_i(x)\|_{L_\infty(R^n)} \leq C_0 \tag{10}$$

$$\|\nabla N_i(x)\|_{L_\infty(R^n)} \leq C_1 \tag{11}$$

$$\tilde{u} \subset H^1(\Omega) \tag{12}$$

where C_0 and C_1 are two constants. $H^\alpha(R)$, $L_2(R)$ and $L_\infty(R)$ refer to Hilbert and Lebesgue spaces, respectively. It is clear that, the first order of differentiability is the minimum condition for the shape functions, not necessarily the best condition.

At this step, we are going to modify the PUM by adding the condition of second order differentiability as the constraint of considering the smooth approximation functions, and separating the approximation functions, as below

$$\left\|\nabla^2 N_i^{sm}(x)\right\|_{L_\infty(R^n)} \leq C_2 \tag{13}$$

$$\tilde{u}^{sm} \subset H^\alpha(\Omega), \quad \alpha = 2$$

$$\tilde{u}^{sh} \subset H^0(\Omega) \tag{14}$$

where the superscripts sm and sh refer to the smooth and sharp shape functions, respectively. Now, we combine the conditions of new PUM and Theorem 2.1 to express the modified MRA and name it as *computational* MRA (CMRA). Here, the criterion (CMRA) for proposing the modified shape (scaling-wavelet) functions is established as follows.

Criterion 2.1: The shape functions $N_i^{sm,sh}$ must hold the following conditions:

$$\begin{aligned}
\text{support}\left(N_i^{sm,sh}\right) &\subset \text{closure}(\Omega_i)\\
\sum_i N_i^{sm,sh} &= 1, \text{on } \Omega\\
\left\|N_i^{sm,sh}\right\|_{L_\infty(R^n)} &\le C_0^{sm,sh}\\
\left\|\nabla N_i^{sm,sh}\right\|_{L_\infty(R^n)} &\le C_1^{sm,sh}\\
\left\|\nabla^2 N_i^{sm}(x)\right\|_{L_\infty(R^n)} &\le C_2\\
\int_{\Omega_k} N^{sm}(x-k)\,dx &= 1\\
\int_{\Omega_k} N^{sh}(x-k)\,dx &= \frac{1}{2}\\
\int_{\Omega_k} N^{sm,sh}(x-k)N^{sm,sh}(x-l)\,dx &= \delta_{\text{Kronec.}}\\
N_i^{sm,sh}(-x) &= N_i^{sm,sh}(x),\\
&\quad(\text{even symmetry}). \qquad (15)
\end{aligned}$$

It should be noted that for simplicity, the common conditions of N^{sm} and N^{sh} have been expressed as $N^{sm,sh}$; but the reader should be aware that each of them is two conditions, i.e., $C_0^{sm,sh}$ means C_0^{sm} and C_0^{sh}, etc. In other words, N_i^{sm} takes the role of scaling function and N_i^{sh} takes the role of wavelet function.

B. Shepard's Method

According to what is proposed by Shepard in scattered data fitting [8]; the shape functions should be in the following form:

$$N_i(x) = \frac{w_i(x)}{P(x)} \qquad (16)$$

where

$$P(x) = \sum_i w_i(x). \qquad (17)$$

In fact, the weighting function w_i is introduced to reach the shape function N_i. The subscript i refers to the weighting function of ith node. So, (16) guarantees that the summation of all shape functions at an arbitrary point is equal to one.

Clearly, many functions can be found that satisfy criterion (15). Here as an example, a shape function for CMRA method is proposed. We use the multiplication of functions $E_i(x) \times D_i(x)$ for the smooth and $A_i(x) \times D_i(x)$ for constructing the sharp weighting functions, i.e., $w_i(x)$. E_i and A_i are usually of the exponential forms [9] to be able to control the decay of the shape function and D_i is usually of the cosine or polynomial type to control its overhanging behavior. We suggest these three functions for 1-D problems as follows:

$$E_i(x) = \exp(-\alpha(x-x_i)^2) \qquad (18)$$
$$D_i(x) = \cos(\beta\pi(x-x_i)) \qquad (19)$$
$$A_i(x) = \exp(-\sigma(|x-x_i|)) \qquad (20)$$

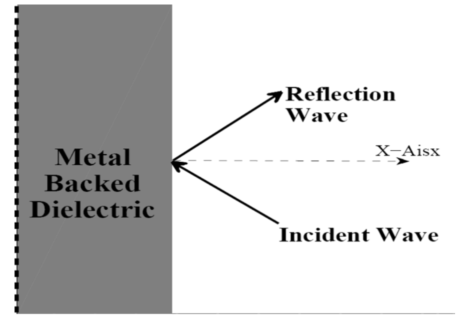


Fig. 1. Problem domain of the incident wave on a metal backed dielectric.

where x_i is the position of scattered node and α, β and σ are positive constants called shape parameters and are determined by some techniques as those in [10], [11] or trial and error method. It is important to note that the presented formulas are for 1-D case where the problem domain has been extended only in one direction, e.g., x direction. The modified formulas can be easily derived for 2- and 3-D cases. So, instead of Daubechies' method a new implementation of shape functions is used.

IV. Analysis of the Incident Wave on Complex Metal Backed Dielectrics

In this section, we are going to check the CMRA and its criterion in a complicated problem. The first subsection introduces the functional principle of the wave equation. Next, we match the shape function on the criterion. Finally, the error and accuracy of the CMRA will be investigated.

A. Wave Equation and its Weak Form

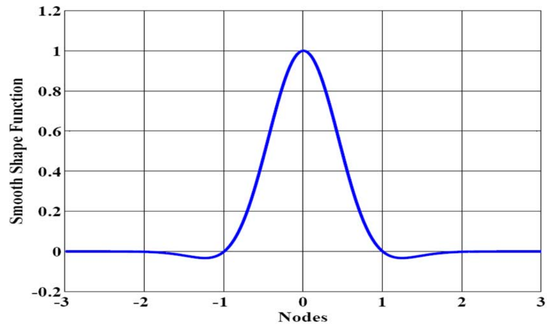
To evaluate the CMRA in electromagnetics, we need to check it in some complicated situations such as the homogenous wave equation in an inhomogeneous medium given below [12]

$$\frac{\partial}{\partial x}\left(f\frac{\partial \phi_z}{\partial x}\right) + k_0^2 g \varphi_z = 0 \qquad (21)$$

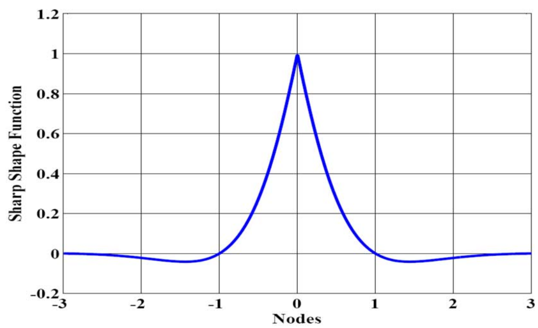
where f and g are coefficient functions, determined later. The problem under consideration is illustrated in Fig. 1, where a uniform plane wave is incident upon an inhomogeneous dielectric slab backed by a conducting plane. The dielectric has thickness $d = 5\lambda_0$, relative permittivity ϵ_r and relative permeability μ_r, both of them can be a function of x. The surrounding medium is free space. We are interested in finding the reflection coefficient. For this example the following complicated medium (shielding medium) parameters are considered for dielectric

$$\mu_r = 2 - 0.1j \qquad (22)$$
$$\epsilon_r = 4 + \mu_r\left(1 - \frac{x}{d}\right)^2. \qquad (23)$$

It is well known that any plane wave can be decomposed into two E_z and H_z-polarized plane waves having only a z-component [12]. Now, let us first identify the domain for analysis. The domain of the problem is obviously semi-infinite, but the meshless method can not be applied to such an unbounded domain.



(a)



(b)

Fig. 2. Proposed (a) smooth and (b) sharp shape functions for wave equation at $f = 3$ GHz.

Therefore, we need to reduce the domain by introducing an artificial boundary, called absorbing boundary, with an appropriate condition. According to [12] this boundary condition can be expressed for E_z as

$$\left[\frac{\partial E_z(x)}{\partial x} + jk_0 \cos\theta E_z(x)\right]_{x=x_0} = 2jk_0 \cos\theta E_0 e^{jk_0 x \cos\theta} \tag{24}$$

where θ is the incident angle and $x_0 = 1.33\,d$. The artificial boundary condition for magnetic polarization can be given by a similar relation. Hence, we consider a general form for this type of boundary condition as

$$\frac{\partial \varphi_z}{\partial x} + h\varphi_z = v\varphi_0 \tag{25}$$

where φ_0 is the wave amplitude, h and v are derived by (24). Now, it is time to separate two different polarizations and express the boundary conditions of each polarization. Then, each of polarizations will be solved under the meshless method based on CMRA.

1) Electric Polarization: For E_z-polarization, it can be shown that the Helmholtz equation (21) governs the electric filed E_z when its parameters are set as the following [12]:

$$\begin{aligned} \varphi_z &= E_z \quad \varphi_0 = E_0 \\ f &= \frac{1}{\mu_r} \quad g = \epsilon_r - \frac{1}{\mu_r}\sin^2\theta. \end{aligned} \tag{26}$$

For the absorbing boundary condition, we have the following parameters for (25):

$$h = jk_0 \cos\theta \quad v = 2jk_0 \cos\theta e^{jk_0 x_0 \cos\theta} \tag{27}$$

and the essential boundary conditions such as Dirichlet and continuity conditions must be enforced as

$$E_z(0) = 0, \quad E_z(d^+) = E_z(d^-). \tag{28}$$

2) Magnetic Polarization: For H_z-polarization, (21) governing the magnetic filed H_z with the following parameters:

$$\begin{aligned} \varphi_z &= H_z \quad \varphi_0 = H_0 \\ f &= \frac{1}{\epsilon_r} \quad g = \mu_r - \frac{1}{\epsilon_r}\sin^2\theta. \end{aligned} \tag{29}$$

For the absorbing boundary condition, similar to previous one we have

$$h = jk_0 \cos\theta \quad v = 2jk_0 \cos\theta e^{jk_0 x_0 \cos\theta} \tag{30}$$

and the continuity condition must be forced as

$$H_z(d^+) = H_z(d^-). \tag{31}$$

There is also a Neumann condition for magnetic polarization as

$$\frac{\partial H_z(0)}{\partial x} = 0. \tag{32}$$

We note that (32) is a natural condition that is automatically satisfied. Using the functional principle, the functional of wave equation is as follows [12]:

$$F(\varphi_z) = \frac{1}{2}\int_\Omega \left[f\left(\frac{\partial \varphi_z}{\partial x}\right)^2 - k_0^2 g \varphi_z^2\right] dx + \frac{1}{2}\left[h\varphi_z^2 - 2v\varphi_z\right]_{x_0}. \tag{33}$$

Let us denote φ_z at ith node by φ_i. Substituting (2) for both smooth and sharp shape functions into (33) and taking derivative with respect to φ_i, gives the system of equations that will be equated to zero to satisfy the stationary point condition (Ritz's method). The systems of equations are constructed as

$$\begin{aligned} [K^{sm}]_{n_1^0 \times n_1^0} [\phi]_{n_1^0 \times 1} &= [B^{sm}]_{n_1^0 \times 1} \\ [K^{sh}]_{n_2^0 \times n_2^0} [\phi]_{n_2^0 \times 1} &= [B^{sh}]_{n_2^0 \times 1}. \end{aligned} \tag{34}$$

The elements of stiffness matrix K, i.e., k_{ij}, and excitation matrix B, i.e., b_i, for both smooth and sharp levels in electric and magnetic polarizations are as follows:

$$k_{ij} = \int_\Omega \left[f \cdot \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right) - k_0^2 g N_i N_j\right] d\Omega \tag{35}$$

$$b_i^{sm,sh} = 0 \tag{36}$$

where n_1^0 and n_2^0 denote the total number of nodes in smooth and sharp subdomains, respectively. The Dirichlet boundary condition is imposed as given by (28) and (31), and third kind boundary condition, i.e., (27) and (30), is imposed as

$$k_{ll}^{sm,sh} = k_{ll}^{sm,sh} + h(x_0) \tag{37}$$

$$b_l^{sm,sh} = v(x_0)\varphi_0 \tag{38}$$

where l denotes the nodes under third kind boundary condition, i.e., the node at $x_0 = 1.33\,d$.

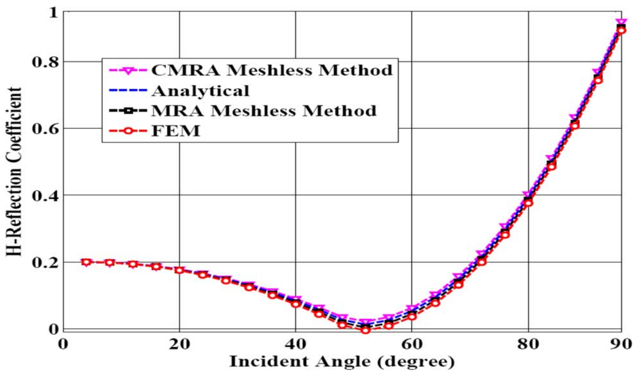


Fig. 3. Reflection coefficient of magnetic polarization at $f = 3$ GHz using different methods.

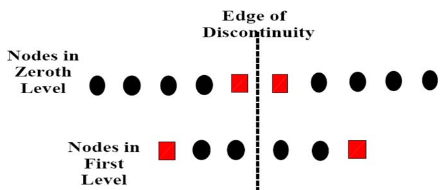


Fig. 4. CMRA discretization used for analyzing the incident wave problem with $J_{\max} = 1$ as the maximum level of resolution.

B. Compatibility on the CMRA Criterion

Let analyze the problem at frequency $f = 3$ GHz. The proposed shape functions matched on criterion 2.1 have been selected in accordance with (18), (19) and (20) as illustrated in Fig. 2 for $\alpha = 1.45$, $\beta = 0.54$ and $\sigma = 1.9$ according to [10], [11]. After calculation of fields, to check the validity of criterion, we are interested in the evaluation of reflection coefficient (power) defined as [12]

$$R = \frac{\varphi_z(d) - \varphi_0 e^{jk_0 d\cos\theta}}{\varphi_0 e^{-jk_0 d\cos\theta}}. \tag{39}$$

Fig. 3 shows the reflection coefficient of magnetic field (due to strict page limitation, the electric reflection coefficient has not been shown). As depicted in the figure, the CMRA method is able to follow the analytical solution with a high accuracy. This fact comes from the separation of entire domain into two smooth and sharp subdomain. In first subdomain we have used the smooth approximation function (14). Then, near the edge of discontinuity the sharp approximation function (14) has been used. This discretization is shown in Fig. 4. The agreement between all methods increases when the incident angle tends to 90°. Substituting the calculated fields into the reflection equation, we evaluate the accuracy of proposed CMRA meshless method. Here, the proposed method has been compared with the analytical solution of reflection coefficient, the FEM and finally, the previous MRA meshless methods. Fig. 5 shows the error with respect to the exact solution per total number of nodes, defined as

$$\text{Error} = \frac{1}{M}\frac{\sum_i^M \left|u_{\text{exact}}^i - u_{\text{numerical}}^i\right|}{\sum_i^M \left|u_{\text{exact}}^i\right|}. \tag{40}$$

For the same degree of error, the CMRA uses less nodes which reduces the time consumption rate. For example, if we are interested in error equal to 0.005, the CMRA reaches this error using 200 nodes, while the MRA and FEM reach this error with 230 and 275 nodes, respectively.

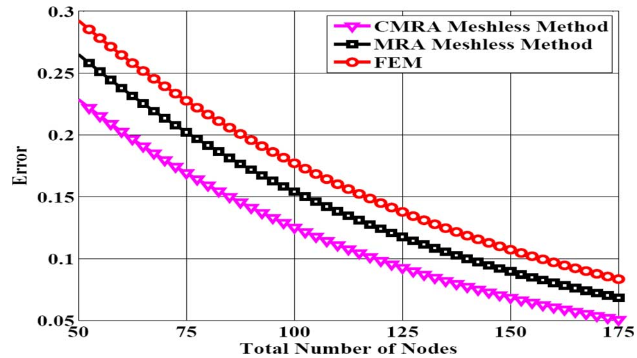


Fig. 5. Error of reflection coefficient with respect to the exact solution, using different methods.

V. Conclusion

In this work, a modified meshless method based on multiresolution analysis has been proposed by which the complex dielectrics can be simulated. This approach helps to overcome the problems of irregularity, computational time and the accuracy of the meshless methods based on wavelet analysis. Some shape functions that are completely matched on the criterion were proposed and testing in a general electromagnetic problems, i.e., the propagation of wave through a complex dielectric, showed extremely good agreement with the exact solution.

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